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Finite Width Corrections to the Nambu Action for the Nielsen-Olesen String

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Abstract

The finite width correction terms to the Nambu action for Nielsen-Olesen strings are calculated. They consist of an extrinsic curvature squared or rigidity term and a new 'twist' term. The extrinsic curvature term prevents cusps forming, rounding them off with a curvature radius of the order of the string width.

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The Nambu action [1] is today ubiquitous in physics. Originally invoked in the context of the dual string model in hadron physics, it is now used as the starting point for theories of fundamental strings [2] and as an approximate description of the motion of Nielsen-Olesen vortex lines [3] in the theory of cosmic strings [4].

What are the corrections to the Nambu action for finite width strings? In the cosmic string theory, the correction terms are almost always tiny, being of order the string width divided by the radius of curvature squared. However generically the motion of a string loop produces 'cusps' [5], and 'kinks' are also frequently generated naturally by string reconnections [6]. Both cusps and kinks propagate with the velocity of light, producing several interesting astrophysical phenomena [7]. However, both are singular points where the curvature radius goes to zero and the Nambu action breaks down. The finite width corrections to the Nambu action become sigmificant at these points.

Recently Polyakov suggested the possibility of adding an extra 'rigidity' term to the Nambu action in a phenomenological description of QCD [8]. The effects of this term have been extensively analysed both classically [9] and quantum mechanically [10]. In particular the 'leading Regge trajectory' Nambu string solutions, doubled lines whose ends rotate at the speed of light, become modified so that the ends are rounded off to a finite curvature radius [9] and move at a slower speed.

In this letter we return to the classical Nielsen-Olesen vortex line and calculate the leading order corrections to the Nambu action. We obtain not only Polyakov's rigidity term with a calculable coefficient but also a new 'twist' term of the same order which has previously been ignored in the literature. The rigidity term in particular has the correct sign to round out 'cusps'.

Our calculation is based on an expansion in the 'width' of the Nielsen-Olesen string w divided by the radius of curvature R of the string trajectory. The basic method we use first appeared in the work of Forster [11] but that work is unfortunately incomplete invoking the strong coupling limit in particular and not proceeding beyond the Nambu action.

Let us consider the 2-dimensional world sheet which is the trajectory of the center of the string (the manifold of zeros of the Higgs field). The world sheet $X^a = X^a(\tau, \sigma)$ is described by two parameters τ and σ (or $\sigma^\mu \equiv (\tau, \sigma)$). For any spacetime point x^a nearer the world sheet than its radius of curvature we can find the nearest point on the worldsheet σ^μ and write

$$x^a = X^a(\sigma) + \rho^A n^a_{(A)}(\sigma) \tag{1}$$

where $n^a_{(A)} (A = 1, 2)$ are two orthonormal vectors perpendicular to the world sheet and ρ^A are the coordinates in those directions.

In order to calculate the effective action for the string motion, we assume that the microscopic structure of string is locally given to zeroth order by the Nielsen-Olesen static vortex solution. We then calculate the first order correction in w/R to the field configuration and substitute this back into the action. Now the calculated corrections to our zeroth order ansatz and thus to the Nambu action become large precisely near 'cusps' and 'kinks' so that strictly speaking treating them as small perturbations is not justified - the resulting value of w/R in the 'corrected' string solutions at the corresponding points is of order unity. However one can clearly see the sign and order of magnitude of the corrections.

The basic idea of the method is to integrate over the ρ coordinates, which parametrise the internal structure of string, to find the action for the string motion in terms of world sheet coordinates σ^μ.

The action we consider is [3]

$$
\begin{aligned}
S &= \int d^4x \;\; L && (2)\\
L &= -\{\mid D\Phi \mid^2 + V(\Phi) + \frac{1}{4} F_{ab}^{\,2}\} && (3)
\end{aligned}
$$

where $D_a \equiv \partial_a + ieA_a$ and $V(\Phi) \equiv \frac{\lambda}{4}(\mid \Phi \mid^2 - \eta^2)^2$. Our metric convention in this paper is $(-,+,+,+)$. In order to perform the integration over ρ, we introduce the curved coordinate system $\varsigma^\alpha = (\sigma^\mu, \rho^A)$. The action (2) is now

$$
S = \int d^2\sigma d^2\rho \sqrt{-G} \;\; L \tag{4}
$$

where L is the same as (3) replacing contractions and derivatives by covariant ones using the metric $G_{\alpha\beta}$. The metric $G_{\alpha\beta}$ in the curved coordinate system is easily calculated from (1) as

$$
ds^2 = \eta_{ab} dx^a dx^b = G_{\alpha\beta} d\varsigma^\alpha d\varsigma^\beta \tag{5}
$$

and is given by

$$
G_{\alpha\beta} = \begin{pmatrix} G_{\mu\nu} & G_{\mu B} \\ G_{A\nu} & G_{AB} \end{pmatrix}
$$

where

$$
\begin{aligned}
G_{\mu\nu} &= \gamma_{\mu\nu} - 2\rho_C K^{(C)}_{\mu\nu} + \rho_C \rho_D \gamma^{\rho\sigma} K^{(C)}_{\mu\rho} K^{(D)}_{\nu\sigma} + \rho^C \rho_C \omega_\mu \omega_\nu \\
G_{\mu B} &= G_{B\mu} = \rho^C \epsilon_{CB} \omega_\mu && (6)\\
G_{AB} &= \delta_{AB}
\end{aligned}
$$

$\gamma_{\mu\nu} \equiv \partial_\mu X \cdot \partial_\nu X$ is the intrinsic metric on the world sheet, the extrinsic curvatures for the two normals $n_{(A)}$ are $K^{(A)}_{\mu\nu} \equiv -\partial_\mu n^{(A)} \cdot \partial_\nu X$ and the 'twist' is $\omega_\mu \equiv \partial_\mu n^{(1)} \cdot n^{(2)}$, where a dot denotes an inner product for Minkowski vectors. Note that we have used the completeness relation $\eta^{ab} = \partial_\mu X^a \partial_\nu X^b \gamma^{\mu\nu} + n^a_{(A)} n^b_{(A)}$. We use the summation convention for repeated indices throughout.

Because of the curvature of the string, we expect the Nielsen-Olesen static vortex solution is also slightly modified as

$$\Phi = [\phi_S(\rho) + \phi_1(\sigma,\rho)] \ \exp\{ \ i[\theta_S(\rho) + \theta_1(\sigma,\rho)]\} \tag{7}$$

$$A_\alpha = (\ 0 \ , \ A_{S,B}(\rho) + A_{1,B}(\sigma,\rho) \ ) \tag{8}$$

where $(\phi_S \exp[i\theta_S], \ A_{S,B})$ is the Nielsen-Olesen static vortex solution and the variables with subscript 1 are the perturbations.

Assuming that the derivatives in the tangential directions on the world sheet are smaller by $O(w/R)$ than those in the normal directions and expanding the action (4) with respect to ρ up to the second order, we find

$$S = \int d^2\sigma d^2\rho \sqrt{-\gamma} \ (1 + J_1 + J_2)[L_0 + L_1 + L_2] \tag{9}$$

where J's and L's are from $\sqrt{-G}$ and L, respectively, and the subscripts 0, 1 and 2 correspond to the order of perturbation expansion.

The 0-th order term L_0 gives the Nambu action. The first order terms, J_1 and L_1, vanish if the 0-th order solution is assumed. The second order action is now described by three terms, *i.e.*

$$S_2 = S_2^{(1)} + S_2^{(2)} + S_2^{(3)} \tag{10}$$

where

$$
\begin{aligned}
S_2^{(1)} &\equiv \int d^2\sigma d^2\rho\sqrt{-\gamma}\ \ J_2\ L_0 \\
&= \frac{1}{2}\int d^2\sigma\sqrt{-\gamma}\left(K^{(C)}K^{(D)} - K^{(C)}_{\mu\nu}K^{(D)\mu\nu}\right)\int d^2\rho\ \rho^C\rho^D\ L_0 \quad (11)
\end{aligned}
$$

$$
\begin{aligned}
S_2^{(2)} &\equiv \int d^2\sigma d^2\rho\sqrt{-\gamma}\ \ J_1\ L_1 \\
&= -\int d^2\sigma\sqrt{-\gamma}\ K^{(C)}\int d^2\rho\ \rho^C\ L_1 \quad (12)
\end{aligned}
$$

$$
S_2^{(3)} \equiv \int d^2\sigma d^2\rho\sqrt{-\gamma}\ \ L_2 \quad (13)
$$

The integration over ρ in $S_2^{(1)}$ gives δ^{CD}. Then $S_2^{(1)}$ vanishes because $K^{(C)}K_{(C)} - K^{(C)}_{\mu\nu}K^{\mu\nu}_{(C)} = R^{(2)}$, the Ricci scalar, and the worldsheet integrand is totally divergent in two dimensions.

Taking the variation of S_2, we find the perturbation equations for ϕ_1, θ_1 and $A_{1,B}$. Those equations contain inhomogeneous terms from the variation of $S_2^{(2)}$ which are proportional to the extrinsic curvatures $K^{(C)}$. Hence general solutions can be written as

$$
\begin{aligned}
\phi_1(\sigma,\rho) &= K_{(C)}(\sigma)\phi_1^{(C)}(\rho) + f(\sigma)\ h_\phi(\rho) \quad (14)\\
\theta_1(\sigma,\rho) &= K_{(C)}(\sigma)\theta_1^{(C)}(\rho) + f(\sigma)\ h_\theta(\rho) \quad (15)\\
A_{1,B}(\sigma,\rho) &= K_{(C)}(\sigma)A_{1,B}^{(C)}(\rho) + f(\sigma)\ h_{A,B}(\rho) \quad (16)
\end{aligned}
$$

where $f(\sigma)$ is an arbitrary function and h_ϕ, h_θ and $h_{A,B}$ are the general homogeneous solutions of the perturbation equations. $\phi_1^{(A)}, \theta_1^{(A)}$ and $A_{1,B}^{(C)}$ are particular inhomogeneous solutions, whose explicit forms will be discussed later.

Solving the perturbation equations and inserting the solutions into the

S_2, we find the total effective action as

$$S = -\mu \int d^2\sigma \sqrt{-\gamma} - \frac{1}{\alpha_0} \int d^2\sigma \sqrt{-\gamma} K^{(A)}_{\mu\nu} K^{\mu\nu}_{(A)} - \frac{1}{\beta_0} \int d^2\sigma \sqrt{-\gamma}\ \omega_\mu \omega^\mu \quad (17)$$

where

$$\mu = -\int d^2\rho\ L_0 \quad (18)$$

$$\begin{aligned} \frac{1}{\alpha_0} &= -\frac{1}{2} \int d^2\rho \delta O \delta \\ &\equiv \frac{1}{2} \int d^2\rho\ [(\partial_A \phi_1^{(C)})^2 + \phi_S^2 (\partial_A \theta_1^{(C)} + e A_{1,A}^{(C)})^2 + \phi_1^{(C)2} (\partial_A \theta_S + e A_{S,A})^2 \\ &+ (\phi_1^{(C)})^2 V''(\phi_S) + 4\phi_S \phi_1^{(C)} (\partial_A \theta_S + e A_{S,A})(\partial_A \theta_1^{(C)} + e A_{1,A}^{(C)}) + \frac{1}{4} F_{1,AB}^{(C)\,2}] \end{aligned} \quad (19)$$

$$\frac{1}{\beta_0} = \frac{1}{2} \int d^2\rho\ \rho^2 [(\partial_A \phi_S)^2 + \phi_S^2 (\partial_A \theta_S + e A_{S,A})^2 + \frac{1}{2} F_{S,AB}^{\,2}] \quad (20)$$

Here $\delta = (\phi_1, \theta_1, A_{1,B})$ and O is the operator occurring in the equation for small fluctuations about a static straight string. As Nielsen-Olesen strings are stable [12] O is negative definite and α_0 is positive. β_0 is obviously positive. The homogeneous solutions do not give any contribution to the effective action (17).

The perturbation equations for the inhomogeneous above solutions can be reduced to a single ordinary differential equation as follows:
Assuming the following functional forms

$$A_{S,B} = \frac{[a_S(\rho) - 1]}{e\rho^2} (-\rho_2, \rho_1) \quad (21)$$

$$\phi_1^{(A)} = \frac{f_1(\rho)}{\rho} (\rho_1, \rho_2) \quad (22)$$

$$A_{1,B}^{(A)} = \frac{a_{1,B}}{\rho} (\rho_1, \rho_2) \qquad \text{with } a_{1,B} = \frac{a_1(\rho)}{\rho} (-\rho_2, \rho_1) \quad (23)$$

$$\partial_B \theta_1^{(A)} \quad + \quad eA_{1,B}^{(A)} = \frac{c_{1,B}}{\rho}\ (\rho_1, \rho_2) \ \text{ with } \ c_{1,B} = \frac{c_1(\rho)}{\rho}\ (-\rho_2, \rho_1) \tag{24}$$

the perturbation equations can be reduced to

$$\left\{\ \frac{1}{\rho}\frac{d}{d\rho}(\rho\frac{d}{d\rho}) \quad + \quad \frac{3a_S^2 - 1}{\rho^2} - V''(\phi_S)\ \right\}\ f_1(\rho) \ = \ \phi_S' + \frac{2\phi_S a_S^2}{\rho} \tag{25}$$

$$a_1(\rho) \ = \ \frac{1}{e\rho} \quad \int \quad d\rho\rho a_S' \quad \text{and} \quad c_1(\rho) \ = \ a_S[1 - \frac{2f_1}{\rho\phi_S}] \tag{26}$$

The origin of the extrinsic curvature squared correction term is easily understood. If one bends a Nielsen-Olesen string, the minimum energy solution is slightly different from the straight string solution. Any perturbation of the straight string solution produces a higher energy per unit length, and that is exactly what happens here. The effect of this term can be seen from the results of [9] where the cusps at the end of the rotating doubled line solution are rounded off to a radius of curvature of the order of $1/\sqrt{\alpha_0\mu}$, which is of the order of the width of the string.

Our new 'twist' term is slightly more subtle. Equation (1) only defines the ρ coordinates up to a local O(2) rotation of the normal vectors. The 'twist' ω_μ is not however invariant under local O(2) rotations - it changes by $\partial_\mu\psi$ where ψ is the rotation angle. Now in the case of a static string for example ω_μ is only a function of σ and (locally) can always be set to zero. In fact it can be seen from the above action that doing so will result in the configuration of least energy. However the string's motion will generally produce a nonzero ω_μ which cannot be gauged away.

Let us first understand this in terms of the fields making up the string.

Imagine a static Nielsen-Olesen string constrained to lie in a helix, and in the minimum energy configuration i.e. with $\omega_\mu = 0$. Now consider the closed line integral of the gauge field down the centre of the helix, out radially, and back up the length of the helix to enter the helix radially again. This is a gauge invariant quantity and measures the number of turns of the helix through the loop times the unit of magnetic flux carried by the string. Now upon evolution with the Nambu equations (see below) the helix becomes a line. If the flux through the loop has *not* changed then the configuration is clearly not just a locally boosted 'static straight string' since it has a net flux winding around it. Conversely if the flux *has* changed then by Faraday's law there must be a nonzero electric field induced along the string axis. In this case too it is not a locally boosted 'static straight string'. Thus we see that the Nambu trajectory inevitably takes a string out of the 'untwisted' internal ground state. In reality the twisted string can of course lose energy by radiating field excitations, but we ignore energy loss processes here. We expect them to be suppressed by the oscillation frequency divided by the mass of the higgs or gauge particles.

In order to see the same thing from the 'string' point of view, consider the oscillating helix solution of the Nambu equations. In the orthonormal gauge this is described by two constant parameters Ω and α (or $\beta \equiv \sqrt{1-\alpha^2}$) as

$$X^a = (\tau, \alpha\sigma, \frac{\beta}{\Omega}\cos(\Omega\sigma)\cos(\Omega\tau), \frac{\beta}{\Omega}\sin(\Omega\sigma)\cos(\Omega\tau)) \qquad (27)$$

Explicit calculation shows that ω_μ can be gauged to zero in the limits of $\beta \to 0$ and $\alpha \to 0$, which correspond to a striaght string and an oscillating loop respectively. However in general ω_μ cannot be gauged away, as may be seen by calculating the curl $\epsilon^{\mu\nu}\partial_\mu\omega_\nu$ which is nonzero. The magnitude of ω_μ is in general comparable to the extrinsic curvature.

ω_μ *can* be gauged away for strings moving in any spatial plane - one of the normals can be chosen to be in the third spatial direction. Thus only the rigidity term is important. In particular, there is a static loop solution with radius $\sim O(1/\sqrt{\lambda}\eta)$ [9] since the coefficients α_0 and β_0 are of order $O(\lambda)$ when $e^2 \sim \lambda$. Unfortunately (or fortunately for the galaxy formation scenario in cosmic string theory) this radius is of the same order of the string width, hence we cannot answer at least by the present rough estimation the question of whether this static loop solution can really exist and makes the universe string-dominated.

In this letter we have calculated the higher-order correction terms to the Nambu action taking into account the width of string. We have found not only the rigidity term (the extrinsic curvature squared), which prevents cusp formation, but also a new 'twist' term. The details will be published elsewhere, including the numerical calculations of the coefficients $1/\alpha_0$ and $1/\beta_0$ for various coupling constants.

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Erratum to:
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Finite Width Corrections to the Nambu Action for the Nielsen-Olesen String

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1. Page 2, line 2, delete ' The rigidity...cusps.'

2. Page 6, Equation 17, change '$-\frac{1}{\alpha_0}$' to '$+\frac{1}{\alpha_0}$'.

3. Page 7, Line 7, delete 'Any perturbation ....of the string.' and replace with 'The true energy per unit length is lower than what one obtains from the zeroth order solution. The sign of our K^2 term is actually *opposite* to that assumed elsewhere in the literature [9],[10] and corresponds to a *negative* rigidity. Nevertheless the argument of [9] still applies and prohibits cusps because the action diverges for solutions 'close' to those with cusps. Cusps are presumably rounded off on a scale $\approx w$.'

4. Page 9, line 6, delete 'In particular...string-dominated' and replace by 'It would be interesting to find solutions to our action (17).'